Chapter 1
Sequences and Series

Section 1.1
Arithmetic sequences

An arithmetic sequence is a sequence in which the difference between consecutive terms is constant. We can obtain the next term by adding the same number each time.

Consider the sequence

$\{2, 6, 10, 14, 18, ...\}$

The difference between consecutive terms is 4.

The difference between consecutive terms does not need to be positive.

Consider the sequence

$\{25, 16, 7, -2, -11, ...\}$

The difference between consecutive terms is –9.

The n^{th} term of the arithmetic sequence is given by:

$$u_n = u_1 + (n-1)d$$

u_n is the n^{th} term of the arithmetic sequence

u_1 is the first term of the arithmetic sequence

d is the common difference $(= u_n - u_{n-1})$

Arithmetic sequences are also referred to as arithmetic progressions.

Example

Given the arithmetic sequence $\{4, 7, 10, 13, ...\}$, find the 99th term.

Solution:

We need to find d and u_1 first:

$u_1 = 4$, since 4 is the first term.

For d, take any pair of consecutive terms, such as the first and second terms.

$$\begin{aligned} d &= u_n - u_{n-1} \\ &= u_2 - u_1 \\ &= 7 - 4 \\ &= 3 \end{aligned}$$

Using the formula,

$$\begin{aligned} u_{99} &= u_1 + (99-1)d \\ &= 4 + (99-1)3 \\ &= 298 \end{aligned}$$

Example

Given the 3rd term and 7th term of an arithmetic sequence as –4 and 16 respectively, find the 20th term.

Solution:

Again, we need to find d and u_1 first.

Using the formula,

$$\begin{aligned} u_3 &= u_1 + (3-1)d \\ u_7 &= u_1 + (7-1)d \end{aligned} \Rightarrow \begin{aligned} -4 &= u_1 + 2d \\ 16 &= u_1 + 6d \end{aligned}$$

Then we use the elimination method by subtracting the second equation from the first equation:

$$\begin{aligned} -20 &= -4d \\ d &= 5 \end{aligned}$$

Put $d = 5$ into either one of the equations. We will use the first equation in this case.

$$\begin{aligned} -4 &= u_1 + 2(5) \\ u_1 &= -14 \end{aligned}$$

Therefore, the 20th term is:

$$\begin{aligned} u_{20} &= u_1 + (20-1)d \\ &= -14 + 19(5) \\ &= 81 \end{aligned}$$

Sometimes, we may not need to use the formula:

Example

Given the first three terms of an arithmetic sequence a, 6, b and the first three terms of another arithmetic sequence –2, a, b, find a and b.

Solution:

Since we are given the consecutive terms of a sequence, we can use d, the common difference, to solve this question.

From the first sequence, the common difference, d_1, can be expressed as

$d_1 = 6 - a = b - 6$

$\Rightarrow a + b = 12$

From the second sequence, the common difference, d_2, can be expressed as

$d_2 = a - (-2) = b - a$

$\Rightarrow 2a - b = -2$

We can solve for a and b by elimination. Adding the two equations gives:

$3a = 10$

$a = \frac{10}{3}$

Then we put $a = \frac{10}{3}$ into the first equation and obtain:

$\frac{10}{3} + b = 12$

$b = \frac{26}{3}$

Example

A plant has an initial height of 20 cm, and it grows by 3 cm every week. What will be the height of the plant after eight weeks?

Solution:

Since the increase in height is a constant value, we know that this is an arithmetic sequence with $u_1 = 20$ and $d = 3$.

Note that we must calculate u_9 rather than u_8 since we need to find the height after eight weeks, with u_1 being after zero weeks. Therefore,

$u_9 = 20 + (9 - 1)(3)$

$= 44$

Exercise 1.1

Section A

1. Find the value of x in each of the following arithmetic sequences.

(a) $\{1, 3, 5, x, 9, ...\}$

(b) $\{-20, -12, -4, x, 12, ...\}$

(c) $\left\{\frac{3}{5}, x, \frac{1}{5}, 0, -\frac{1}{5}, ...\right\}$

(d) $\{x, -4.2, -2.2, -0.2, 1.8, ...\}$

(e) $\{5.6, 3.4, x, -1, -3.2, ...\}$

(f) $\left\{x, 2\frac{1}{2}, 1\frac{3}{8}, \frac{1}{4}, -\frac{7}{8}, ...\right\}$

2. Show that the following sequences are arithmetic.

(a) $\{2, 7, 12, ...\}$

(b) $\{-14, -8, -2, ...\}$

(c) $\{1.12, 2.8, 4.48, ...\}$

(d) $\left\{-\frac{2}{9}, \frac{2}{9}, \frac{2}{3}, ...\right\}$

(e) $\left\{-3\frac{5}{12}, -\frac{11}{12}, 1\frac{7}{12}, ...\right\}$

(f) $\{x^2 - 3x, 4x^2 - x, 7x^2 + x, ...\}$

3. Determine the fourth term in each of the following arithmetic sequences.

(a) $\{1, 5, 9, ...\}$

(b) $\{250, 1{,}375, 2{,}500, ...\}$

(c) $\left\{-\frac{1}{2}, -\frac{1}{4}, 0, ...\right\}$

(d) $\{-4.5, 1, 6.5, ...\}$

(e) $\{-4\pi, -\pi, 2\pi, ...\}$

(f) $\{-7y^2 + 9, -4y^2 + 13, -y^2 + 17, ...\}$

(g) $\{12, 2, -8, ...\}$

(h) $\{3.5, -2, -7.5, ...\}$

(i) $\left\{1\frac{6}{13}, \frac{7}{13}, -\frac{5}{13}, ...\right\}$

(j) $\{-5.26, -8.4, -11.54, ...\}$

(k) $\{4a + 8b, 2a + 6b, 4b, ...\}$

(l) $\{5x^2 + 2x, 8x^2 - x, 11x^2 - 4x, ...\}$

4. Find the general term of the arithmetic sequence in which the first term is -4 and the common difference is 3. Hence find the sixth term of the sequence.

5. Find the general term of the arithmetic sequence in which the first term is 6 and the common difference is -4. Hence find the twelfth term of the sequence.

6. Find the number of terms in each of the following arithmetic sequences:

(a) $\{3, 9, 15, ..., 93\}$

(b) $\{-28, -25, -22, ..., 59\}$

Section B

1. Find the general term of the each of following arithmetic sequences:

   (a) $\{1, 5, 9, ...\}$
   (b) $\{1{,}255, 155, -945, ...\}$
   (c) $\left\{2\frac{1}{2}, 1\frac{3}{8}, \frac{1}{4}, -\frac{7}{8}, ...\right\}$
   (d) $\{-5.24, -8.4, -11.56, ...\}$
   (e) $\{-7.5\pi, -4.5\pi, -1.5\pi, ...\}$
   (f) $\{-7y^2+5, -4y^2+7, -y^2+9, ...\}$

2. If the general term of a sequence is $u_n = -32 + (n-1)\times(-5)$, what is its 50^{th} term?

3. Find the 23^{rd} term of the arithmetic sequence $\{3, 9, 15, ...\}$.

4. Find the 100^{th} term of the arithmetic sequence $\{27, 12, -3, ...\}$.

5. In an arithmetic sequence, $u_4 = 30$ and $u_8 = 50$. Find the common difference (d) and the general term of this sequence.

6. In an arithmetic sequence, $u_5 = 125$ and $u_{17} = 29$.

   (a) Find the common difference.

   (b) Find the general term of the sequence.

   (c) Find u_{2009}.

7. The general term of a sequence is $u_n = -100 + (n-1)\times(9)$. If $u_k = 125$, find the value of k.

8. If the first three terms of an arithmetic sequence are 5, 21, 37 and the k^{th} term is 421, find the value of k.

9. Find the values of x such that the following sequences form arithmetic progressions:

   (a) $\{..., 2x, 2x+6, 3x, ...\}$
   (b) $\{..., 5-x, 2x+1, 4x, ...\}$

10. Determine the value of x if $-12, x, 6$ form an arithmetic progression.

11. Determine the common difference for $u_n = \frac{6}{7}n$.

12. Find the value of n if the n^{th} term is the first negative term in each of the following arithmetic sequences:

    (a) $\{27, 23, 19, ...\}$

    (b) $\{87, 81, 75, ...\}$

13. A doctor has been working at the same hospital for 10 years. His initial salary was \$32,000 per month, with a \$2,500 increase each year. What is his monthly salary during his 10^{th} year of employment?

14. Joyce goes running every day, and can run 100 m more each day than the day before. If she runs 2,000 m on Wednesday, what distance can she run on the Saturday of the following week?

Section 1.2
Arithmetic series

In the last section, we learnt how to find a particular term of an arithmetic sequence. In this section, we will find the sum of the first n terms of an arithmetic sequence. This is known as an arithmetic series.

Consider the sequence

{1, 5, 9, 13, …}

We can find the sum of the first n terms of the sequence by adding them together:

$1 + 5 = 6$
$1 + 5 + 9 = 15$
$1 + 5 + 9 + 13 = 28...$

However, this is very time consuming if there is a large number of terms.

We need a formula to evaluate the sum. First, we denote the sum of the first n terms of an arithmetic sequence as S_n. Then we can express it in the following two ways:

$S_n = u_1 + u_2 + ... + u_n$
$S_n = u_n + u_{n-1} + ... + u_1$

By adding the two equations, we obtain:

$$2S_n = (u_1 + u_n) + (u_2 + u_{n-1}) + \ldots + (u_n + u_1)$$
$$2S_n = (u_1 + u_1 + (n-1)d) + (u_1 + d + u_1 + (n-2)d) + \ldots + (u_1 + (n-1)d + u_1)$$
$$2S_n = (2u_1 + (n-1)d) + (2u_1 + (n-1)d) + \ldots + (2u_1 + (n-1)d)$$
$$2S_n = n(2u_1 + (n-1)d)$$
$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

We can use this formula to find the sum of the first n terms of an arithmetic sequence:

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

S_n is the sum of the first n terms of the arithmetic sequence

u_1 is the first term of the arithmetic sequence

d is the common difference $(= u_n - u_{n-1})$

Looking closely at the above formula, we can use the last term instead of the common difference to obtain the sum:

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$
$$= \frac{n}{2}(u_1 + u_1 + (n-1)d)$$
$$= \frac{n}{2}(u_1 + u_n)$$

Therefore we get this formula:

$$S_n = \frac{n}{2}(u_1 + u_n)$$

S_n is the sum of first n terms of the arithmetic sequence

u_1 is the first term of the arithmetic sequence

u_n is the n^{th} term of the arithmetic sequence