Contents

**Extended level materials recommended for students who intend to study IB Diploma Mathematics at Higher Level.*

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Chapter 1
Arc Lengths and Areas of Sectors

1.1 Radian measure

The radian is a standard unit of angular measure and is commonly used as an alternative to the degree. The relationship between degrees (°) and radians (rad) is as follows:

$$\pi(\text{rad}) = 180^\circ$$

Note that we do not usually use a symbol to indicate radians. If there is no symbol after an angular measure, then it is in radians rather than degrees. Sometimes, the notation "c" is used to represent the radian, so 2^c means an angle of 2 radians.

The conversion between degrees and radians is done as follows:

From degrees to radians: multiply the angle by $\frac{\pi}{180^\circ}$

From radians to degrees: multiply the angle by $\frac{180^\circ}{\pi}$

Here are some examples:

$$30^\circ = 30^\circ \times \frac{\pi}{180^\circ} = \frac{\pi}{6}$$

$$120^\circ = 120^\circ \times \frac{\pi}{180^\circ} = \frac{2\pi}{3}$$

$$\frac{\pi}{4} = \frac{\pi}{4} \times \frac{180^\circ}{\pi} = 45^\circ$$

$$\frac{3\pi}{2} = \frac{3\pi}{2} \times \frac{180^\circ}{\pi} = 270^\circ$$

Example

Express 210° in radians as a fraction of π.

Solution:

$$210^\circ = 210^\circ \times \frac{\pi}{180^\circ}$$
$$= \frac{210\pi}{180}$$
$$= \frac{7\pi}{6}$$

Example

Express $\frac{4\pi}{3}$ radians in degrees.

Solution:

$$\frac{4\pi}{3} = \frac{4\pi}{3} \times \frac{180^\circ}{\pi}$$
$$= 240^\circ$$

Example

Express 120.4° in radians, correct to three significant figures.

Solution:

$$120.4^\circ = 120.4^\circ \times \frac{\pi}{180^\circ}$$
$$= 2.10 \text{ (correct to 3 s.f.)}$$

Note: If the angle is not an integer, we do not need to express it in terms of π.

Example

Express 2 radians in degrees, correct to three significant figures.

Solution:

$$2 = 2 \times \frac{180^\circ}{\pi}$$
$$= 115^\circ \text{ (correct to 3 s.f.)}$$

Exercise 1.1

Section A

1. Express the following angles in degrees.

(a) $\frac{\pi}{2}$ (b) $\frac{5\pi}{2}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

(e) $\frac{\pi}{6}$ (f) $\frac{7\pi}{4}$

2. Express the following angles in degrees, giving your answers to the nearest 0.1°.

(a) 2.1 (b) 4

(c) 1.245 (d) 3.14

3. Express the following angles in radians in terms of π.

(a) 180° (b) 60°

(c) 135° (d) 15°

(e) 210° (f) 315°

4. Express the following angles in radians, giving your answers to one decimal place.

(a) 42° (b) 310°

(c) 91° (d) 208°

Section B

1. Express the following angles in degrees.

(a) $\frac{8\pi}{15}$ (b) $\frac{17\pi}{12}$

(c) $\frac{13\pi}{9}$ (d) $\frac{37\pi}{20}$

2. Express the following angles in degrees, giving your answers to three significant figures.

(a) $\sqrt{3}$ (b) $\frac{2}{\pi}$

(c) $1+\sqrt{2}$ (d) $\frac{7}{\pi}$

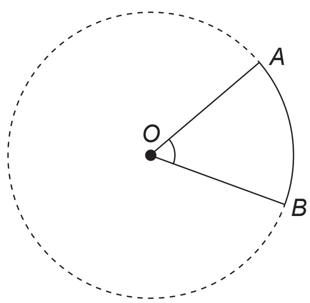
3. Express the following angles in radians, giving your answers to three significant figures.

   (a) 102.01° (b) 31.7°

   (c) 191° (d) 902.444°

1.2 Arc lengths

The circumference is the perimeter of a circle, and an arc is part of the circumference. Consider the following diagram:



The arc *AB* is part of the circumference and the region *OAB* is called a sector. In order to distinguish between the longer arc (the dashed arc) and the shorter arc (the solid arc), we name them the major arc and the minor arc, respectively. Similar terms are also applied to sectors, and these are known as major sectors and minor sectors.

To find the arc length, we first need to find the circumference and then divide it by 360° (this is the arc length for one degree). Then, we multiply the result by θ, where θ (in degrees) is $A\hat{O}B$, to obtain the required arc length.

Arc length $= 2\pi r \times \dfrac{\theta}{360^\circ}$ where r is the radius and θ is the angle (in degrees) at the center.

If the given angle is in radians, we can use the following formula to calculate the arc length:

> Arc length $= r\theta$ where r is the radius of the circle and θ is the angle (in radians) at the center.

The proof is as follows:

Since 360° is equivalent to 2π radians, for θ in radians we have:

$$\begin{aligned}\text{Arc length} &= 2\pi r \times \frac{\theta}{2\pi} \\ &= r\theta\end{aligned}$$

Example

A sector in a circle of radius 6 cm subtends an angle of 70° at the center. Find its arc length, giving your answer in terms of π.

Solution:

$$\begin{aligned}\text{Arc length} &= 2\pi(6) \times \frac{70^\circ}{360^\circ} \\ &= \frac{7\pi}{3}\ \text{cm}\end{aligned}$$

Example

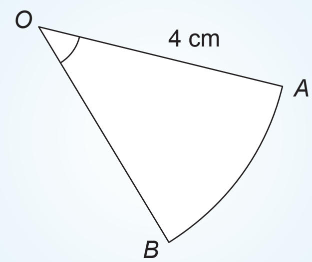
A sector in a circle of radius 6 cm subtends an angle of $\frac{\pi}{3}$ at the center. Find its arc length, giving your answer in terms of π.

Solution:

If the given angle is in radians, we can use the expression $r\theta$ to find the arc length:

$$\begin{aligned}\text{Arc length} &= r\theta \\ &= 6\left(\frac{\pi}{3}\right) \\ &= 2\pi\ \text{cm}\end{aligned}$$

Example



If the length of arc AB is π cm, find the value of $A\hat{O}B$ in radians.

Solution:

Since the angle is required in radians, we have:

Arc length $= r\theta$

$\pi = 4(A\hat{O}B)$

$A\hat{O}B = \frac{\pi}{4}$ rad

Exercise 1.2

Correct your answers to three significant figures and give angles in radians, unless otherwise specified.

Section A

1. An arc AB of a circle with center O and radius 6 cm subtends an angle of $\frac{\pi}{3}$ at O. Find the exact length of arc AB.

2. An arc AB of a circle with center O and radius 10 cm subtends an angle of $\frac{\pi}{4}$ at O. Find the exact lengths of the major arc and the minor arc AB.

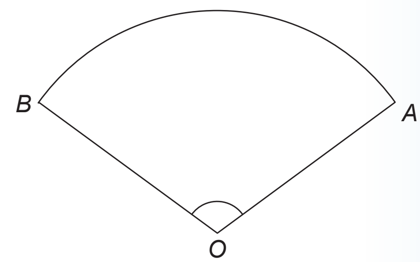
3. Consider an arc AB of a circle with center O. Find the radius of the circle if
   (a) the length of arc AB is 8 cm and it subtends an angle of 2 radians at O;
   (b) the length of arc AB is 12.32 cm and it subtends an angle of 1.54 radians at O.

4. Consider an arc *AB* of a circle with center *O*. Find the angle it subtends at *O* if

   (a) the radius of the circle is 8 cm and the length of arc *AB* is 12 cm;

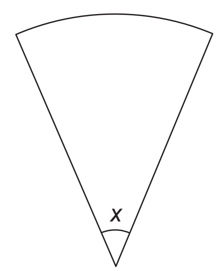
   (b) the radius of the circle is 6 cm and the length of arc *AB* is 4.5π cm.

5. The given diagram shows a sector of a circle of radius $\sqrt{13}$ cm. It subtends an angle of $\frac{3\pi}{5}$ at center *O*. Find the perimeter of this sector.



6. A sector subtending an angle of 1.7 radians is cut from a circle of circumference 15 m. Find the arc length of this sector.

7. The given diagram shows a sector of a circle of radius 8 mm. The perimeter of this sector is 22.4 mm. Find *x*.



8. An arc *AB* of a circle with center *O* and radius 20 cm subtends an angle of *x* at *O*. The major arc *AB* subtends an angle of 7*x* at *O*. Find the exact length of the minor arc *AB*.

9. The given figure shows a sector subtending an angle of 1.8 radians made by a piece of wire. If this piece of wire is used to make a circle, what would be its radius?